STABILITY OF THE TRIVIAL SOLUTION OF A SYSTEM OF TWO LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

(OB USTOICHIVOSTI TRIVIAL'NOGO RESHENIIA SISTEMY DVUKH LINEINYKH DIFFERENTSIAL'NYKH URAVNENII S PERIODICHESKIMI KOEFFITSIENTAMI)

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We propose to find a new estimate of the characteristic constant of Liapunov [1] for the system mentioned in the title when the functions on the secondary diagonal of the coefficient matrix are of definite and the same sign.

1. Let the equations of the first approximation of the perturbed motion of a dynamical system be given in the form

$$\frac{dx_1}{dt} = a_{11}(t)\,x_1 + a_{12}(t)\,x_2, \qquad \frac{dx_2}{dt} = a_{21}(t)\,x_1 + a_{22}(t)\,x_2 \tag{1}$$

where $a_{ij}(t)$ $(i, j = 1, 2)$ are real, integrable, sectionally continuous periodic functions of period ω. We introduce the notation

$$\int_0^\omega a_{11}(t)\,dt = \alpha, \qquad \int_0^\omega a_{22}(t)\,dt = \beta$$

and, without affecting the generality, we assume that $\alpha > \beta$. If $\alpha + \beta > 0$, the unperturbed motion of the dynamical system for which the equation of the first approximation of the perturbed motion has the form (1), is stable by a well-known formula of Liapunov. In the sequel we shall assume that $\alpha + \beta < 0$, which, in view of the fact that $\alpha - \beta > 0$, can be written in the form

$$\beta \leqslant \alpha \leqslant -\beta \qquad (\beta \leqslant 0) \tag{2}$$

Making the change of variables

$$x_1 = x \exp \int_0^t a_{11}(t_1)\,dt_1, \qquad x_2 = y \exp \int_0^t \left[a_{22}(t_1) + \frac{\alpha - \beta}{\omega}\right] dt_1$$

we transform the system (1) to the form

$$\frac{dx}{dt} = r(t)\,y, \qquad \frac{dy}{dt} = q(t)\,x - ay \qquad \left(a = \frac{\alpha - \beta}{\omega} \geqslant 0\right) \tag{3}$$

where

$$r(t) = a_{12}(t) \exp \int_0^t [a_{22}(t_1) - a_{11}(t_1) + a]\, dt_1$$

$$q(t) = a_{21}(t) \exp \int_0^t [a_{11}(t_1) - a_{22}(t_1) - a]\, dt_1$$

The characteristic equation for the system (3) has the form

$$\rho^2 - 2A\rho + e^{-a\omega} = 0 \qquad \left(A = \tfrac{1}{2}\,\mathrm{sp}\, X(\omega) \qquad X(0) = I_2\right),$$

where $X(t)$ is the matrix of the system (3). The characteristic constant A, in the sense of Liapunov [1], can be represented by a converging power series

$$A = A_0 + A_1 + A_2 + \cdots \tag{4}$$

The formulas for the coefficients A_n were derived* in [2].

$$A_0 = \tfrac{1}{2}(1 + e^{\beta - \alpha}), \qquad A_1 = \tfrac{1}{2}\int_0^\omega e^{at_1}[\psi(\omega) - (1 - e^{-a\omega})\psi_1]\, q_1\, dt_1$$

$$A_n = \tfrac{1}{2}\int_0^\omega dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} e^{at_1}(\psi_1 - \psi_2)\ldots \tag{5}$$

$$\ldots e^{at_{n-1}}(\psi_{n-1} - \psi_n)\, e^{at_n}[\psi(\omega) - \psi_1 + \psi_n e^{-a\omega}]\, q_1 \ldots q_n dt_n$$

where

$$\psi(t) = \int_0^t e^{-at_1} r(t_1)\, dt_1, \qquad \psi_i = \psi(t_i), \qquad q_i = q(t_i)$$

By means of an orthogonal transformation with periodic coefficients (see [3]) one can reduce the system (1) to a form in which the functions a_{12} and a_{21} (and hence the functions r and q) are of a definite sign.

In [4], the functions r and q were assumed to be of definite but opposite signs.

2. From here on we shall always assume that r and q are of definite sign on $[0, \omega]$ (furthermore, $r(t)q(t) \neq 0$ for $0 \leqslant t \leqslant \omega$, because otherwise the system (3) can be integrated over intervals).

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* In [2] the constant A stood for sp $X(\omega)/(1 + e^{-a\omega})$.

In this case it follows from (5) that $A_\nu > 0$ ($\nu = 1, 2, \ldots$) and the series (4) is positive-definite.

In [2] it was shown that if $\alpha + \beta \leqslant 0$, the unperturbed motion of the dynamical systemk for which the equation of the first approximation has the form (1), is unstable if $|A| > (e^{-\alpha} + e^{\beta})/2$. Since $A > A_0$ here, it is sufficient to require for instability that $A_0 > (e^{-\alpha} + e^{\beta})/2$, which in view of the first formula of (5) can be written in the form

$$(1 - e^{-\alpha})(1 - e^{\beta}) \geqslant 0$$

Recalling (2) we conclude that if $\alpha > 0$ then the last inequality is satisfied. Hence, we have proved the following theorem.

Theorem. Suppose that in the equations of the first approximation of the perturbed motion (1) the functions a_{12} and a_{21} are of definite sign (i.e. retaining the same sign, they may become zero but in such a way that their product does not vanish identically on $(0, \omega)$ and is of one sign). If, furthermore, the mean value of one of the functions a_{11} or a_{22} is non-negative, then the unperturbed motion is unstable.

Consequence. If in the equation

$$\ddot{x} + s(t)\dot{x} + p(t)x = 0, \qquad (s(t+\omega) = s(t), \quad p(t+\omega) = p(t)) \tag{6}$$

the periodic coefficient $p(t)$ is non-positive for all values of $t (0 \leqslant t \leqslant \omega)$, then the trivial solution of Equation (6) is unstable irrespective of the behavior of the second periodic coefficient $s(t)$.

For the proof it is sufficient to write Equation (6) in the form of the system

$$\frac{dx}{dt} = \dot{x} \qquad \frac{d\dot{x}}{dt} = -p(t)x - s(t)\dot{x}$$

and to apply the established theorem.

3. The case that remains to be considered is the one when $\beta \leqslant \alpha < 0$. In this case the undisturbed motion will be stable if

$$A < \frac{1}{2}(e^{-\alpha} + e^{\beta}) \tag{7}$$

and it will be unstable if the inequality sign is reversed [2].

In [2] there were proved certain inequalities which can be written for the terms of the series (4) in the following way:

$$\frac{A_n}{A_{n-1}} < 2\frac{A_1}{n}, \qquad \frac{A_n}{A_{n-1}} < e^{\alpha-\beta}\frac{A_1}{n^2} \qquad (n \geqslant 2) \tag{8}$$

4. Because of the first of the inequalities (8) with $n = 2, 3, \ldots,$

we have that

$$A_2 < A_1^2, \qquad A_3 < \frac{2}{3} A_1 A_2 < \frac{2}{3} A_1^3, \ldots, A_n < \frac{2^{n-1}}{n!} A_1^n$$

Hence, we find that

$$A < A_0 + A_1 + \tfrac{1}{2} \sum_{n=2}^{\infty} \frac{2^n}{n!} A_1^n = \tfrac{1}{2} e^{\beta-\alpha} + \tfrac{1}{2} e^{2A_1}$$

Under the condition that

$$\tfrac{1}{2} e^{\beta-\alpha} + \tfrac{1}{2} e^{2A_1} \leqslant \tfrac{1}{2} (e^{-\alpha} + e^{\beta}) \text{ или } A_1 \leqslant \tfrac{1}{2} \ln (e^{-\alpha} + e^{\beta} - e^{\beta-\alpha})$$

the inequality (7) will hold, and, hence, the unperturbed motion will be stable.*

5. Let us now apply to the above estimates the second inequality of (8)

$$A_2 < \tfrac{1}{4} e^{\alpha-\beta} A_1^2, \quad A_3 < \tfrac{1}{9} e^{\alpha-\beta} A_1 A_2 < \tfrac{1}{36} e^{2(\alpha-\beta)} A_1^3, \ldots, A_n < \frac{1}{(n!)^2} e^{(n-1)(\alpha-\beta)} A_1^n$$

We thus obtain

$$A < A_0 + A_1 + \sum_{n=2}^{\infty} \frac{1}{(n!)^2} e^{(n-1)(\alpha-\beta)} A_1^n = \tfrac{1}{2} (1 - e^{\beta-\alpha}) + e^{\beta-\alpha} I_0 (2 \sqrt{e^{\alpha-\beta} A_1})$$

where $I_0(z)$ is a Bessel function of an imaginary argument. If

$$\tfrac{1}{2} (1 - e^{\beta-\alpha}) + e^{\beta-\alpha} I_0 (2 \sqrt{e^{\alpha-\beta} A_1}) \leqslant \tfrac{1}{2} (e^{-\alpha} + e^{\beta})$$

i.e.

$$I_0 (2 \sqrt{e^{\alpha-\beta} A_1}) \leqslant \tfrac{1}{2} (1 + e^{\alpha} + e^{-\beta} - e^{\alpha-\beta}) \tag{10}$$

then the unperturbed motion is stable.** The inequality (10) is preferable over (9) when $\alpha = \beta < \ln 4$.

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* In the last formula on p. 136 of [2] the denominator of the logarithmic term should be dropped.

** In Equation (11.5) of [2] the right-hand side should be written in the form

$$\tfrac{1}{2} (1 + e^{\alpha} + e^{-\beta} - e^{\alpha-\beta})$$

and the index of the exponent of the left-hand side in the form $\alpha - \beta$.

If we begin the estimates of the terms of the series (4) with A_3, we find that if either one of the following inequalities

$$A_2 \leqslant \frac{1}{e^{2A_1} - 1 - 2A_1} [(e^{-\alpha} - 1)(1 - e^{\beta}) - 2A_1] A_1^2 \qquad (11)$$

or

$$A_2 \leqslant \frac{[(e^{-\alpha} - 1)(1 - e^{\beta}) - 2A_1] e^{2(\alpha-\beta)} A_1^2}{8 [I_0(2\sqrt{e^{\alpha-\beta} A_1}) - 1 - e^{\alpha-\beta} A_1]} \qquad (12)$$

holds then the undisturbed motion is stable.

6. In [4] there was proved an inequality which for the terms of the series (4) can be written in the form

$$\frac{A_{n+1}}{A_n} \leqslant \frac{n}{n+1} \frac{A_n}{A_{n-1}} \qquad (n \geqslant 1) \qquad (13)$$

If we set $n = 1, 2, 3$ we obtain, respectively

$$\frac{A_2}{A_1} \leqslant \frac{1}{2} \frac{A_1}{A_0}, \qquad \frac{A_3}{A_2} \leqslant \frac{2}{3} \frac{A_2}{A_1}, \qquad \frac{A_4}{A_3} \leqslant \frac{3}{4} \frac{A_3}{A_2}, \ \ldots$$

Making use of the products of the last inequalities, we obtain the result for the sum of the series (4):

$$A \leqslant A_0 + A_1 + \frac{1}{2} \frac{A_1^2}{A_0} + \frac{1}{3} \frac{A_1 A_2}{A_0} + \frac{1}{4} \frac{A_1 A_3}{A_0} + \ldots =$$

$$= A_0 + \frac{2}{3} A_1 + \frac{1}{6} \frac{A_1^2}{A_0} + \frac{A_1}{A_0} \left[\frac{1}{3} A_0 + \frac{1}{3} A_1 + \frac{1}{3} A_2 + \frac{1}{4} A_3 + \ldots\right] \qquad (14)$$

The sum of the series within the brackets is less than $A/3$, and hence we have

$$A\left(1 - \frac{1}{3} \frac{A_1}{A_0}\right) < A_0 + \frac{2}{3} A_1 + \frac{1}{6} \frac{A_1^2}{A_0}$$

If $A_1 > 3A_0$, then the last inequality is obviously valid. If, however, $A_1 < 3A_0$, then

$$A < \frac{6A_0^2 + 4A_1 A_0 + A_1^2}{6A_0 - 2A_1}$$

and under the condition

$$\frac{6A_0^2 + 4A_1 A_0 + A_1^2}{3A_0 - A_1} \leqslant e^{-\alpha} + e^{\beta} \qquad (15)$$

the unperturbed motion is stable.

Let us now select from the right-hand side of the inequality (14) all terms up to and including A_2

$$A \leqslant A_0 + \frac{3}{4} A_1 + \frac{1}{4} \frac{A_1^2}{A_0} + \frac{1}{12} \frac{A_1 A_2}{A_0} + \frac{A_1}{A_0} \left[\frac{1}{4} A_0 + \frac{1}{4} A_1 + \frac{1}{4} A_2 + \frac{1}{4} A_3 + \ldots \right]$$

We thus obtain

$$A\left(1 - \frac{1}{4} \frac{A_1}{A_0}\right) < A_0 + \frac{3}{4} A_1 + \frac{1}{4} \frac{A_1^2}{A_0} + \frac{1}{12} \frac{A_1 A_2}{A_0} \tag{16}$$

which, obviously, will be true if $A_1 > 4A_0$. If $A_1 < 4A_0$, then, under the condition that

$$\frac{12A_0^2 + 9A_1A_0 + 3A_1^2 + A_1A_2}{4A_0 - A_1} < \frac{3}{2}(e^{-\alpha} + e^{\beta})$$

the unperturbed motion is stable.

7. In regard to the sufficient conditions for instability, it can be said that they can be found more simply. Indeed, $A > A_0 + A_1$ and, therefore, if $A_0 + A_1 > e^{-\alpha} + e^{\beta})/2$ we have also that $A > (e^{-\alpha} + e^{\beta})/2$. Hence, if the inequality

$$A_1 \geqslant \frac{1}{2}(e^{-\alpha} - 1)(1 - e^{\beta}) \tag{17}$$

is valid, then the unperturbed motion is unstable.

Starting with the inequality $A > A_0 + A_1 + A_2$, we find that if

$$A_2 \geqslant \frac{1}{2}(e^{-\alpha} - 1)(1 - e^{\beta}) - A_1 \tag{18}$$

then the unperturbed motion is unstable.

It is now obvious how one can obtain other sufficient conditions for stability or instability by the indicated method.

BIBLIOGRAPHY

1. Liapunov, A.M., Sur une série dans la théorie des équations différentielles linéaires du second ordre à coefficients périodiques. *Zap. Akad. nauk po fiz.-matem. otd. 8-ia seriia* Vol. 13, No. 2, pp. 1-70, 1902; *Sobr. soch.* Vol. 2, pp. 410-472, 1956.

2. Starzhinskii, V.M., Ob ustoichivosti trivial'nogo resheniia differentsial'nogo uravneniia vtorogo poriadka s periodicheskimi koeffitsientami (Stability of the trivial solution of a second-order differential equation with periodic coefficients). *Inzh. sb.* Vol. 18, pp. 119-136, 1954.

3. Starzhinskii, V.M., Zamechanie k issledovaniiu periodicheskikh dvizhenii (Remark on the investigation of periodic motions). *PMM* Vol. 19, No. 1, pp. 119-120, 1955.

4. Starzhinskii, V.M., O metode Liapunova otsenki kharakteristicheskoi postoiannoi (Liapunov's method for estimating the characteristic constant). *Izv. Akad. Nauk SSSR, OTN, seriia mekh. i mashinostr.* No. 4, 1959.

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